

# Rocket performance

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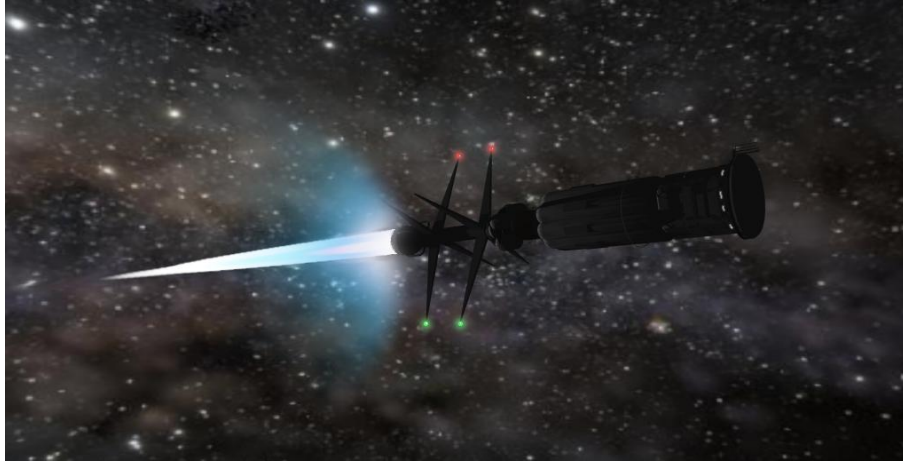


Image from Steve Bowers

A rocket is a device that moves in one direction by expelling part of itself in the other direction. Due to the conservation of momentum, if a rocket at rest shoots out a mass  $m$  of propellant with velocity  $v_{ex}$  (the exhaust velocity), the remainder of the rocket, with mass  $M$ , will end up moving at a speed of

$$v = v_{ex} \frac{m}{M}$$

From this, it is immediately obvious that the more propellant a rocket carries, and the faster it expels its propellant away, the faster the rocket can go.

## $\Delta v$

One complication is that when a rocket expels some of its propellant, this will not just accelerate the rocket's structure and payload, but also all the propellant that has not been used yet. If the total amount of propellant is small compared to the mass of the payload, this will not make much of a difference, but the final velocity will be limited to much less than the exhaust velocity. If the rocket wants to go nearly as fast or faster than it is shooting away its propellant, it will need to carry a mass of propellant much larger than its payload mass. This, in turn, lessens the speed at which initial burns of the propellant can get the rocket going since these initial burns are pushing the mass of the extra propellant along with the rocket. The total amount by which a rocket can change its speed is given by

$$\Delta v = v_{ex} \ln \frac{M_0}{M}$$

where  $M_0$  is the initial mass of the rocket (including payload, structure and propellant) and  $M$  is the total mass remaining after it has burnt its propellant. The function  $\ln$  is the natural logarithm, or logarithm to the base  $e \approx 2.718$ . Thus, for a rocket to get moving at the speed of its exhaust velocity, it

will need to have an initial mass 2.718 times larger than its final mass. To go twice as fast as its exhaust velocity, its initial mass will need to be 7.389 times larger than its final mass.

The  $\Delta v$  found for  $M$  with empty propellant tanks is the total amount by which a rocket can change its velocity. If it uses all of its  $\Delta v$  to speed up, it will have no propellant left to slow down again, and will be left forever drifting at its burnout velocity. Rockets without auxiliary forms of propulsion need to carefully budget their  $\Delta v$  for their mission.

## Force and acceleration

The force being exerted is the rate at which the momentum changes with time. For a rocket, this is

$$F = m'v_{ex}$$

where  $m'$  is the rate at which the rocket is losing propellant mass (for example,  $m'$  might be measured in kilograms of propellant per second). The acceleration a rocket will experience for a given force is inversely proportional to the rocket's mass,

$$a = \frac{F}{M}$$

Since the mass of a rocket will change as it spends propellant, the rocket can accelerate faster near the end of its mission than near the beginning.

As an example, a rocket that shoots out 10 kg of propellant per second at an exhaust velocity of 5000 m/s (typical of high performance chemical fuel rockets) yields 50 000 N of force. If the rocket has a mass of 10 000 kg, it will be accelerated at 5 m/s<sup>2</sup>, or about half the acceleration due to gravity on the surface of old Earth.

## Power

It takes energy to get propellant moving. The kinetic energy of a mass  $m$  moving at a speed  $v_{ex}$  is

$$E = \frac{1}{2}mv_{ex}^2$$

Power is the rate at which energy is gained or lost, so the power required to expel propellant at a mass flow of  $m'$  and an exhaust velocity of  $v_{ex}$  is

$$P = \frac{1}{2}m'v_{ex}^2$$

In addition, the force,  $F = m'v_{ex}$ , can be substituted into this relationship for power, to relate the power to the force and exhaust velocity,

$$P = \frac{1}{2}Fv_{ex}$$

or the mass flow rate,

$$P = \frac{F^2}{2m'}$$

## Efficiency

All this assumes that the rocket shoots its propellant straight backwards, and that all the power goes into moving the propellant and not into residual heat in the exhaust or waste heat in the rocket. If some of the propellant goes to the side (perhaps the exhaust is shot out in a cone instead of straight back, so that some of the propellant goes a bit to one side or the other) it will not get quite as much thrust. This is the nozzle momentum efficiency. If the propellant exhaust stream exits the rocket hotter than it started out in the propellant tanks, it will take more power. This is the nozzle energy efficiency. With well engineered rockets, these are not usually large effects. For example, a rocket that shoots its propellant backward in a cone that is  $\frac{1}{10}$  as wide as it is long will still have 99.75 % as much thrust as if it shot its propellant straight back. Rockets using solid nozzles have efficiencies as high as 70 %. Magnetic nozzles can give efficiencies of up to 85 % for thermal plasmas, or higher for some athermal plasmas. If the nozzle energy efficiency is denoted  $\zeta$  and the nozzle momentum efficiency is denoted  $\xi$  then

$$\begin{aligned} F &= \xi m' v_{ex} \\ P &= \frac{m' v_{ex}}{2\zeta} \\ P &= \frac{F v_{ex}}{2\zeta\xi} \\ P &= \frac{F^2 v_{ex}}{2\zeta\xi^2 m'} \end{aligned}$$

## Temperature

Often, a rocket works by heating its propellant until it is a gas or plasma, and then allowing the hot, high pressure propellant to expand away from the rocket in the backward direction, pushing the rocket forward. As the propellant expands, its temperature drops until most of its initial thermal energy has been turned into kinetic energy. The exhaust velocity can be calculated from the propellant's temperature and the mass of a given quantity of propellant particles.

Given the temperature  $T$  in kelvin, the efficiency  $\zeta$  of the rocket, and the molar mass  $\rho$  (amount of mass in one mole, and  $1 \text{ mol} \approx 6.022 \times 10^{23}$  particles) of the expelled propellant, the exhaust velocity will be

$$v_{ex} = \sqrt{\frac{3\zeta RT}{\rho}}$$

where  $R$  is the gas constant,  $R = 8.314 \text{ J mol}^{-1} \text{ K}^{-1}$ . The molar mass of a type of atom is called its atomic mass, and is usually listed in descriptions of that atom or isotope. The value for  $R$  given above assumes that  $\rho$  is being measured in kg/mol, which has to be accounted for given that molar masses are often listed in g/mol.

Often, the temperature will be expressed as an energy  $\tau$ . Given that a single particle in the spent propellant has a rest mass of  $\epsilon$  expressed as energy, then

$$v_{ex} = c \sqrt{\frac{3\zeta\tau}{\epsilon}}$$

where  $c$  is the speed of light in vacuum,  $c = 299\,792\,458 \text{ m/s}$ . For example, one atom of normal hydrogen has an energy of 939 MeV due to its rest mass. If a fusion reactor heats hydrogen up to a temperature of 50 keV, and the hydrogen plasma is shot out through a magnetic nozzle at 90 % efficiency to form a

rocket plume, the exhaust velocity will be

$$\begin{aligned} v_{ex} &= c \sqrt{\frac{3 \times 0.90 \times 0.050 \text{ MeV}}{939 \text{ MeV}}} \\ &= 0.012 c \\ &= 3.6 \times 10^6 \text{ m/s} \end{aligned}$$

Very commonly, a rocket's exhaust will be made of more than one kind of particle. In this case, the weighted average mass of all the particles in the gas has to be used.

For example, a fusion plasma might consist of 80 % helium-3 with a mass of 2809 MeV per particle or 0.003 016 kg/mol, and 20 % deuterium with a mass of 1876 MeV per particle or 0.002 014 kg/mol. The effective mass per particle is then

$$\begin{aligned} m &= 0.80 \times 2809 \text{ MeV} + 0.20 \times 1876 \text{ MeV} \\ &= 2622 \text{ MeV} \end{aligned}$$

Thus, the effective molar mass is

$$\begin{aligned} \rho &= 0.80 \times 0.003\,016 \text{ kg/mol} + 0.20 \times 0.002\,014 \text{ kg/mol} \\ &= 0.002\,816 \text{ kg/mol} \end{aligned}$$

## Relativistic rockets

The previous descriptions assume that  $v_{ex}$  and  $\Delta v$  are both much less than the speed of light in vacuum. When this assumption breaks down, the analysis will need to be modified. First, a quantity termed  $\beta$  needs to be calculated

$$\beta = \frac{v_{ex}}{c}$$

The total change in rapidity  $\eta$  available to the rocket is

$$\Delta\eta = \beta \ln \frac{M_0}{M}$$

Unlike velocities, at relativistic speeds rapidities do add. So a rocket can accelerate up to any rapidity given by its  $\Delta\eta$ , and use any remaining  $\Delta\eta$  to slow down or change direction. For a given rapidity, the velocity is

$$v = c \tanh \eta$$

where  $\tanh$  is the hyperbolic tangent,

$$\tanh \eta = \frac{e^\eta - e^{-\eta}}{e^\eta + e^{-\eta}}$$

and the time dilation factor for the rocket is

$$\gamma = \cosh \eta$$

where  $\cosh$  is the hyperbolic cosine,

$$\cosh \eta = \frac{e^\eta + e^{-\eta}}{2}$$

If the total energy  $E$  of a particle in the exhaust is known, including its rest mass-energy, then quantity  $\gamma_{ex}$  can be calculated

$$\gamma_{ex} = \frac{E}{\epsilon}$$

or by knowing its kinetic energy,  $E_k$

$$\gamma_{ex} = 1 + \frac{E_k}{\epsilon}$$

where, as before,  $\epsilon$  is the rest mass-energy of one particle. Once  $\gamma_{ex}$  is known  $\beta$  can be calculated.

$$\beta = \sqrt{1 - \frac{1}{\gamma_{ex}^2}}$$

For a photon rocket, this method works perfectly well, by making  $\beta = 1$ , that is,  $v_{ex} = c$ .

For example, protons and antiprotons have rest masses of 938 MeV each. When they annihilate each other, they produce 1877 MeV of energy. This annihilation produces, on average, 1.5  $\pi^+$  mesons, 1.5  $\pi^-$  mesons, and 2.0  $\pi^0$  mesons. A  $\pi^+$  or  $\pi^-$  meson has a rest mass of 140 MeV and a  $\pi^0$  has a rest mass of 135 MeV. In total, then, the rest mass the proton-antiproton annihilation produces is

$$\begin{aligned} m &= 1.5 \times 140 \text{ MeV} + 1.5 \times 140 \text{ MeV} + 2.0 \times 135 \text{ MeV} \\ &= 690 \text{ MeV} \end{aligned}$$

Therefore,

$$\begin{aligned} \gamma_{ex} &= \frac{1877 \text{ MeV}}{690 \text{ MeV}} \\ &= 2.72 \end{aligned}$$

Thus  $\beta$  can be solved for,

$$\begin{aligned} \beta &= \sqrt{1 - \frac{1}{2.72^2}} \\ &= 0.93 \end{aligned}$$

In actual practice, only the  $\pi^+$  and  $\pi^-$  mesons contribute to thrust (each  $\pi^0$  meson immediately decays into two gamma rays, which cannot be magnetically deflected). This reduces the available  $\Delta\eta$  by a factor of

$$\begin{aligned} \Delta\eta &= \frac{1.5 \times 140 \text{ MeV} + 1.5 \times 140 \text{ MeV}}{690 \text{ MeV}} \\ &= 61 \% \end{aligned}$$